Multifractal Discrete Stochastic Volatility

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Introduction

- Modelling financial data is essentially done through the study of the log-return.
- Regime switching processes become a powerful tool to model, forecast and interpret financial data.
- These models try to mimic some specifics characteristics (called *stylized facts*) presented by log-returns.
- The MSM (Calvet and Fisher (2004)) has proved to be a strong competitor of GARCH class of models for modelling volatility of returns.

Motivation

- MSM model the volatility of log-return by a high-dimensional Markov chain obtained by the product of lower-dimensional Markov chains (with 2 states).
- MSM is parsimonious.
- Nice interpretation of MSM : Financial volatility is impacted by the arrival of news in the market whose impact persist for varying periods of time
- MSM allow to capture three types of features view in the literature as distinct : low-frequency variations, intermediate-frequency dynamics and high-frequency switches.

Motivation

- MSM does not help to reproduce some stylized facts such as leverage effect.
- MSM is based on the product of 2-states Markov chains.
- Several recent studies (Hansen and Huang (2016); Liu and Maheu (2018), etc.) showed that instead of modeling uniquely log-returns, adding realized measures helps to improve the information about the current level of volatility.

What do we do?

We propose a new process (Multifractal Discrete Stochastic Volatility, MDSV) that :

- generalize MSM and other related models ,
- slow the decay of the auto-correlation function at finite lags (a stylized fact) ,
- take into account leverage effect ,
- jointly model financial log-returns and realized volatilities ,
- allow to capture low-frequency variations, intermediate-frequency dynamics and high-frequency switches such as MSM ,
- can be interpreted as a multi-component stochastic volatility model .

Outlines

MDSV : Model presentation

- Model :

$$\begin{cases} r_t & = & \sqrt{V_t}\,\epsilon_t\,, \\ \log RV_t & = & \xi + \varphi \log(V_t) + \delta_1\epsilon_t + \delta_2(\epsilon_t^2 - 1) + \gamma\varepsilon_t\,, \end{cases}$$

where $\xi \in \mathbb{R}$, $\varphi \in \mathbb{R}$, $\delta_1 \in \mathbb{R}$, $\delta_2 \in \mathbb{R}$, and $\gamma \in (0, \infty)$ are parameters, and ϵ_t and ε_t are mutually and serially independent innovation processes with mean 0 and variance 1.

- Volatility process V_t is express as the product of two components :
  - A persistent component M_t ,
  - A component L_t capturing leverage effect .

$$V_t = M_t\,L_t\,.$$

MDSV : Model presentation

- **Persistent component :**

The component $\{M_t\}$, MDSV(N, K), is constructed as the product of N independent Markov chains of dimension K :

$$M_t = \frac{\sigma^2}{m} \prod_{i=1}^{N} M_t^{(i)} ,$$

with $M_t^{(i)}$ Markov chains with state space $\boldsymbol{\nu}^{(i)}$, stationary distribution $\boldsymbol{\pi}^{(i)}$ and the transition matrix $P^{(i)}$ defined by

$$P^{(i)} = \phi_i \mathbf{I}_K + (1 - \phi_i) \mathbf{1}_K \boldsymbol{\pi}^{(i)} ,$$

where $\phi_i \in (0, 1)$, and $\phi_i = a^{b^{i-1}}$, $\forall\, i = 1, 2, \ldots, N$.

MDSV : Model presentation

Parsimony is reach by setting some restrictions. After all, the component $\{M_t\}$ depends on five parameters (independent of N):

$$\sigma^2 \in (0,\infty)\,, \quad \nu_0 \in (0,1)\,, \quad a \in (0,1)\,, \quad b \in (1,\infty)\,, \quad \omega \in (0,1)\,.$$

Remark: Although the state space is of dimension K^N, it contains $N(K-1)+1$ distinct values.

- **Leverage component :**

$$L_t = \prod_{i=1}^{N_L} L_t^{(i)} \text{ avec } L_t^{(i)} = 1 + l_i \frac{|r_{t-i}|}{\sqrt{L_{t-i}}} \mathbf{1}_{\{r_{t-i}<0\}}\,,$$

where

$$l_i = \theta_l^{i-1} l_1 \text{ and } l_1 > 0\,,\ \theta_l \in [0,1]\,.$$

Some remarks

Theorem (Properties of MDSV) :

- For $k \geq 1$, we have
$\mathbb{C}ov[M_{t+k}, M_t] = \sigma^4 \left(\prod_{i=1}^{N} \left(1 + \left(A^{K-1} - 1\right) \phi_i^k\right) - 1 \right)$,
where A is a constant greater than 1 .
- $\mathbf{P}^k - \mathbf{\Pi} = \mathcal{O}\left(k^{K-2} a^k\right)$.
- The distribution of the time that the chain $\{M_t\}$ spends in one of the (non-extremal) $N(K-1)+1$ distinct values of the state space is not geometric (semi-Makov property).

Remark 1 :

- MDSV generalize MSM and like models.
- MDSV can be interpreted as a multi-component stochastic volatility model.

Performance measurement tools

- Fitting (in-sample) :

$$\begin{aligned} \text{AIC} &= \mathcal{L} - k\,, \\ \text{BIC} &= \mathcal{L} - \frac{k}{2}\log(n)\,, \end{aligned}$$

- Forecasting (out-of-sample) :

$$\begin{aligned} \text{RMSFE}(h) &= \sqrt{\frac{1}{T-h+1}\sum_{t=0}^{T-h}\left[\left(\frac{1}{h}\sum_{i=1}^{h}\hat{x}_{t+i} - \frac{1}{h}\sum_{i=1}^{h}x_{t+i}\right)^2\right]}\,, \\ \text{MAFE}(h) &= \frac{1}{T-h+1}\sum_{t=0}^{T-h}\left|\frac{1}{h}\sum_{i=1}^{h}\hat{x}_{t+i} - \frac{1}{h}\sum_{i=1}^{h}x_{t+i}\right|\,, \end{aligned}$$

Data

- 4 daily financial time series available on the Oxford-Man Institute of Quantitative Finance's web site :
  - S&P 500 (*Standard & Poor's* 500)
  - NASDAQ 100 (*National Association of Securities Dealers Automated Quotation* 100)
  - FTSE 100 (*Financial Times Stock Exchange* 100)
  - NIKKEI 225 (*Nihon Keizai Shinbun* 225)
- Period : from January 1^{st}, 2000 to December, 31^{st}, 2019.

Fitting performance

Table: Comparison of fits

	Benchmarks			MDSV		
	Real EGARCH	MS-RV $K = 2$	MS-RV $K = 4$	(N, K) (3, 10)	(N, K) (6, 3)	(N, K) (10, 2)
Np	10	7	19	12	12	12
	S&P 500 ($n = 5016$)					
$\mathcal{L}$	-6786.1	-8324.1	-7402.2	**-6673.8**	-6687.9	-6701.0
AIC	-6796.1	-8331.1	-7421.2	**-6685.8**	-6699.9	-6713.0
BIC	-6828.7	-8354.0	-7483.2	**-6724.9**	-6739.0	-6752.2
	NASDAQ 100 ($n = 5012$)					
$\mathcal{L}$	-8661.3	-10624.0	-9802.2	**-8546.9**	-8568.3	-8590.3
AIC	-8671.3	-10631.0	-9821.2	**-8558.9**	-8580.3	-8602.3
BIC	-8703.9	-10653.8	-9883.2	**-8598.0**	-8619.4	-8641.5
	FTSE 100 ($n = 5042$)					
$\mathcal{L}$	-7809.9	-8918.2	-8302.4	**-7761.4**	-7804.8	-7789.4
AIC	-7819.9	-8925.2	-8321.4	**-7773.4**	-7816.8	-7801.4
BIC	-7852.5	-8948.0	-8383.4	**-7812.6**	-7855.9	-7840.6
	NIKKEI 225 ($n = 4865$)					
$\mathcal{L}$	-9004.3	-10586.1	-10000.5	**-8839.2**	-8863.2	-8877.5
AIC	-9014.3	-10593.1	-10019.5	**-8851.2**	-8875.2	-8889.5
BIC	-9046.8	-10615.8	-10081.1	**-8890.1**	-8914.1	-8928.4

Np : Number of parameters, log-lik : log-likelihood, AIC : Akaike Information Criteria, BIC : Bayesian Information Criteria, n : Sample size. Highest values are in bold.

Forecasting performance

Table: Forecasting

horizon(h)	RMSFE 1	RMSFE 5	RMSFE 25	RMSFE 100	MAFE 1	MAFE 5	MAFE 25	MAFE 100
S&P 500								
Real EGARCH	0.45	0.43	0.45	0.44	0.21	0.22	0.32	0.39
MS-RV(2)	0.52	0.50	0.52	0.56	0.30	0.33	0.46	0.50
MS-RV(4)	0.63	0.77	0.91	1.67	0.26	0.30	0.48	0.97
MDSV(3, 10)	0.45	0.42	0.41	0.37	0.21	0.21	0.27	0.32
MDSV(6, 3)	0.44	0.41	0.40	0.35	0.20	0.19	0.23	**0.25**
MDSV(10, 2)	**0.43**	**0.41**	**0.39**	**0.35**	**0.19**	**0.19**	**0.23**	0.25
NASDAQ 100								
Real EGARCH	0.56	0.51	0.50	0.51	0.27	0.28	0.36	0.45
MS-RV(2)	0.86	5.44	4.09	5.20	0.55	0.87	1.07	1.13
MS-RV(4)	0.69	0.70	0.84	0.74	0.34	0.36	0.46	0.59
MDSV(3, 10)	0.55	0.49	0.44	0.41	0.25	0.25	0.30	0.33
MDSV(6, 3)	**0.55**	**0.49**	**0.44**	**0.39**	0.25	0.25	0.30	0.31
MDSV(10, 2)	0.55	0.49	0.44	0.39	**0.24**	**0.25**	**0.28**	**0.29**

Conclusion

- Expand regime switching model by adding a more general model (generalization of MSM and other related models).
- MDSV take into account more stylized effect like leverage effect.
- MDSV allow to capture simultaneously low, intermediate and high frequencies dynamics.
- MDSV has a large state space and is parsimonious.
- MDSV has the ability to generate a high degree of volatility persistence.
- MDSV has a semi-Markov property on the distinct values of the state space.
- MDSV can be interpreted as a multi-component stochastic volatility model.
- MDSV can be integrate into a joint model (log-returns and realized variance) framework.
- We provide a package available on GitHub :
  `https://github.com/Abdoulhaki/MDSV`

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